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A Sixth-Order Explicit Single-Step Nonlinear Numerical Method

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ABSTRACT

In this work, a sixth-order explicit single step non-linear method for solving initial value problems whose solutions possess singularities is constructed. The stability and convergence properties of the constructed method are investigated. Also, the associated local truncation error is presented. Using some standard test problems, the accuracy and efficiency of the method are established by its implementation and the results obtained are compared with those discussed in the literature.

Keywords: nonlinear; singularities; ordinary differential equations; first order; initial value problems

INTRODUCTION

The assumption that the solution of a first order ordinary differential equation

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \quad x \in [x_0, X], \quad \mathbf{y}(x_0) = \boldsymbol{\eta} \qquad (1)$$

can be represented locally by a polynomial is the basis for many numerical methods. However, when a given initial value problem or its theoretical solution $y(t)$ is known to posses a singularity, then representing $y(x)$ in the neighbourhood of the singularity by a polynomial will be particularly inappropriate [3, 7, 8]. This is true as general linear multistep and Runge-Kutta type methods usually produce very poor solutions around singularity points [7, 3, 6, 8]. Rational functions, however, are more appropriate for representing functions close to singularities than polynomials.

The use of rational functions in the local representation of theoretical solutions in the neighbourhood of singularity is the underlying assumption in the construction of nonlinear schemes [18, 13, 12]. Interestingly, this approach is now gaining popularity as several methods are now being constructed using under this assumption [14, 4, 1, 10]. The works of the authors in [12, 13, 16, 18, 1] showed that solution around singularity point are well approximated by this approach. In this work, a sixth-order explicit single-step nonlinear method for solving 1 is presented.

CONSTRUCTION OF SCHEME

In this section, we present the construction process of the proposed method It is assumed that the solution $y(x)$ can be locally represented by a rational function $R(x)$ of the form

$$R(x) = \frac{\sum_{i=0}^{m} a_i x^i}{\sum_{j=0}^{n} b_j x^j} \qquad (2)$$

The choice of $m = 5, n = 1$ in (2) is peculiar to this work. In order to construct the associated nonlinear scheme, (2) is required to satisfy the following:

$$\left.\begin{aligned} R(x_{n+j}) &= y_{n+j}, \quad j = 0,1 \\ r^{(i)}(x_{n+j}) &= y_{n+j}^{(i)}, \quad j = 0, \quad i = 1,2,3,4,5,6. \end{aligned}\right\} \qquad (3)$$

Substituting for expressions and simplifying (3) yields

$$y_n(x) = \frac{a_5 x_n^5 + a_4 x_n^4 + a_3 x_n^3 + a_2 x_n^2 + a_1 x_n + a_0}{b_0 + x_n} \qquad (4)$$

$$y_{n+1}(x) = \frac{a_5(h+x_n)^5 + a_4(h+x_n)^4 + a_3(h+x_n)^3 + a_2(h+x_n)^2 + a_1(h+x_n) + a_0}{b_0 + h + x_n} \qquad (5)$$

$$y_n'(x) = \frac{5a_5 x_n^4 + 4a_4 x_n^3 + 3a_3 x_n^2 + 2a_2 x_n + a_1}{b_0 + x_n} - \frac{a_5 x_n^5 + a_4 x_n^4 + a_3 x_n^3 + a_2 x_n^2 + a_1 x_n + a_0}{(b_0 + x_n)^2} \qquad (6)$$

$$y_n^{(2)}(x) = 2(3(a_4 - a_5 b_0)x_n + a_5 b_0^2 - a_4 b_0 + 6a_5 x_n^2 + a_3) + \frac{a_0 - b_0(b_0(b_0(a_5 b_0^2 - a_4 b_0 + a_3) - a_2) + a_1)}{(b_0 + x_n)^3} \quad (7)$$

$$y_n^{(3)}(x) = 6\left(\frac{b_0(b_0(b_0(a_5 b_0^2 - a_4 b_0 + a_3) - a_2) + a_1) - a_0}{(b_0 + x_n)^4} - a_5 b_0 + 4a_5 x_n + a_4\right) \quad (8)$$

$$y_n^{(4)}(x) = 24\left(\frac{a_0 - b_0(b_0(b_0(a_5 b_0^2 - a_4 b_0 + a_3) - a_2) + a_1)}{(b_0 + x_n)^5} + a_5\right) \quad (9)$$

$$y_n^{(5)}(x) = -\frac{120\left(a_0 - b_0(b_0(b_0(a_5 b_0^2 - a_4 b_0 + a_3) - a_2) + a_1)\right)}{(b_0 + x_n)^6} \quad (10)$$

$$y_n^{(6)}(x) = \frac{720\left(a_0 - b_0(b_0(b_0(a_5 b_0^2 - a_4 b_0 + a_3) - a_2) + a_1)\right)}{(b_0 + x_n)^7} \quad (11)$$

Eliminating the undetermined coefficients $a_0, a_1, a_2, a_3, a_4, a_5, b_1$ in (4-11) results in the scheme

$$y_{n+1} = \frac{h^5 \left(y_n^{(5)}\right)^2}{4\left(30 y_n^{(5)} - 5h y_n^{(6)}\right)} + \frac{1}{24} h^4 y_n^{(4)} + \frac{1}{6} h^3 y_n^{(3)} + \frac{1}{2} h^2 y_{n''} + h y_{n'} + y_n \quad (12)$$

The resulting method (12) is explicit, self-starting and nonlinear. We shall refer to (12) as ***NLM5*** which is the method proposed in this work.

LOCAL TRUNCATION ERROR AND ABSOLUTE STABILITY OF CONSTRUCTED METHOD

In this section, the local truncation error (lte) and the absolute stability properties of the new method proposed in this work are considered.

Local Truncation Error

- ***Definition (Local Truncation Error)***

The local truncation error T_{n+1} at x_{n+1} of the general explicit single-step method is given as

$$T_{n+1} = y(x_{n+1}) - y(x_n) - h\phi(x_n, y(x_n), h) \quad (13)$$

where, $y(x_n)$ is the theoretical solution.

Using the above definition, it follows that the local truncation error of the constructed one step methods can be written as

$$T_{n+1} = y(x_{n+1}) - y_{n+1} \quad (14)$$

- ***Definition (Order)***

A numerical method is said to be of order p if p is the largest integer for which $T_{n+1} = \mathcal{O}(h^{p+1})$ for every n and $p \geq 1$.

Following the above definition, the local truncation error of the method constructed in this work is obtained as the residual when y_{n+1} is replaced by $y(x_{n+1})$. Below is the local truncation error for the method constructed in this work.

$$T_{n+1} = \frac{h^7 \left(y^{(6)}\right)^2}{4320 y^{(5)}} \quad (15)$$

Consistency

A scheme is said to be consistent if the difference equation of the integrating formula exactly approximates the differential equation it intends to solve as the step size approaches zero. In order to establish the consistency property of the constructed methods, it is sufficient to show that

$$\lim_{h \to 0} \frac{y_{n+1} - y_n}{h} = 0$$

Now, let the Right Hand Side (RHS) of (12) be denoted as $\phi_n(x)$. Using the above denotation, we have that

$$\lim_{h \to 0} \frac{\phi_n(x) - y_n}{h} = 0 \quad (16)$$

the above indicates that the constructed scheme satisfy the consistency property.

- ***Definition (Convergence)***

A numerical method is said to be of convergent if it is consistent and has an order $p > 1$.

From the above, it is clear that the proposed scheme is convergent.

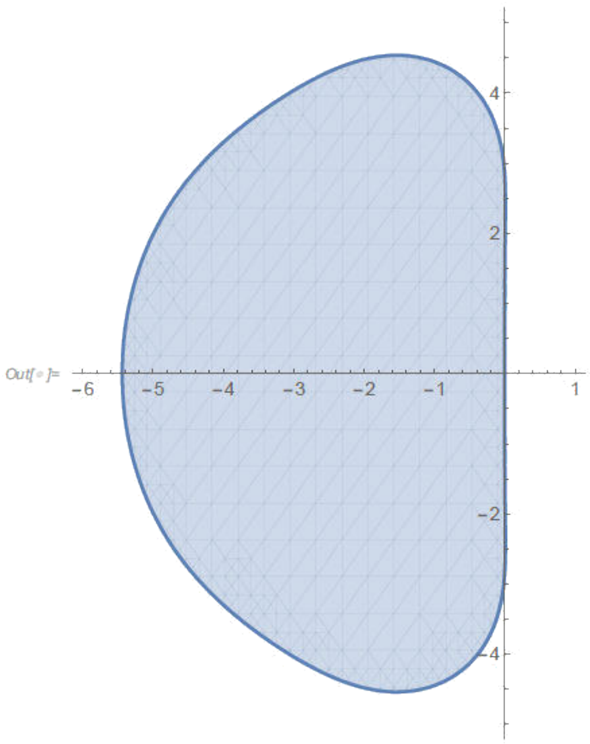

FIGURE 1: Absolute stability region of (12)

Stability

To get the stability behaviour of the constructed scheme, the scheme was implemented on the standard test problem

$$y' = \lambda y, \quad Re(\lambda) < 0$$

and the stability polynomial $R(z) = \frac{y_{n+1}}{y_n}$, where $z = \lambda h$ is obtained. The stability polynomials is obtained as follows:

$$-\frac{(z(z(z(z(z+10)+60)+240)+600)+720)y_n}{120(z-6)}$$

From Figure (1), the stability interval of the method is obtained as $[-5.4, 0]$. This is a wider range compared with those of existing method.

NUMERICAL EXAMPLES

The first problem considered in this work is the nonlinear initial value problem

$$y' = 1 + y^2; \qquad y(0) = 1 \qquad (17)$$

whose theoretical solution is given as

$$y(x) = \tan\left(x + \frac{\pi}{4}\right) \qquad (18)$$

This problem possess a singularity at $x = \frac{\pi}{4}$. Here, the absolute errors of the results obtained by our proposed method are first compared with those of the derivative-free methods proposed in [12] and [16] as shown in Figure 2. A comparison of the maximum absolute error obtained by the proposed method against those respectively produced by the methods of the authors in [14, 4, 12, 10, 8, 16] is a also presented in Figure 3.

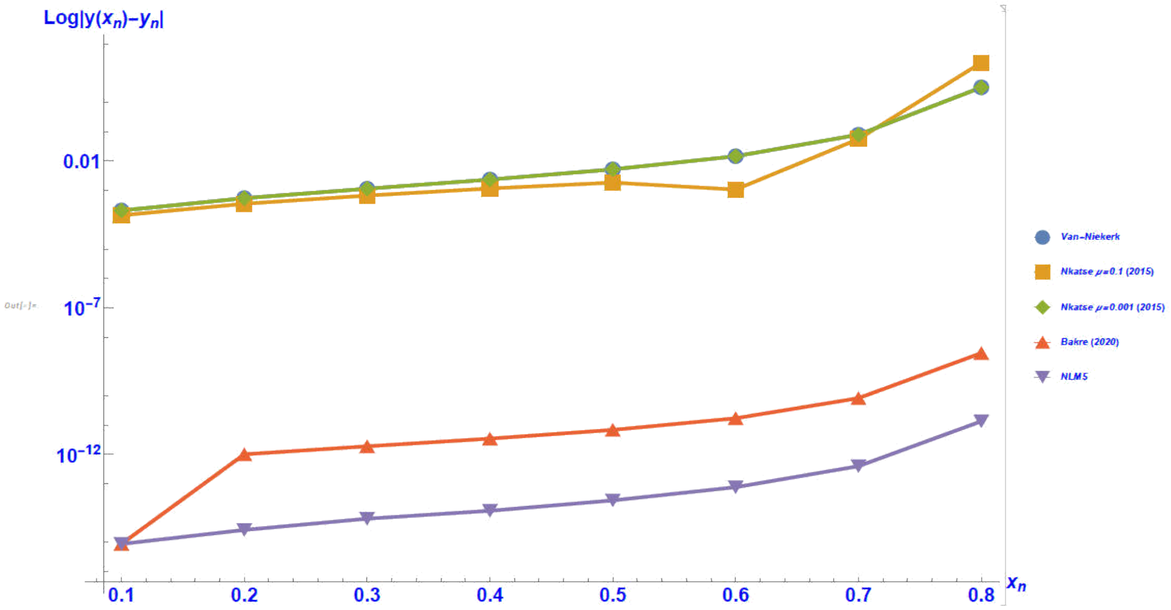

FIGURE 2: Logarithm of absolute errors for the solutions of Problem 1 with step–size h = 0.01.

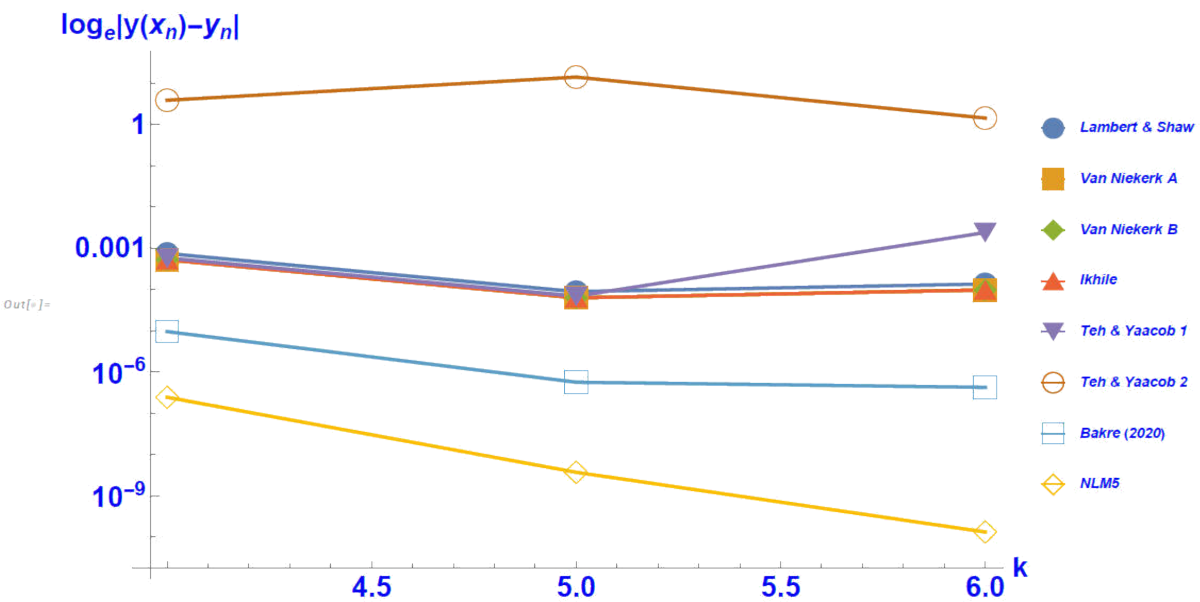

FIGURE 3: Log plot of maximum absolute errors for Problem 1 as a function of the step–size $h = \frac{0.8}{2^k}$, $k = 4(1)6$.

Problem 2

The second test problem considered is given as

$$y' = y^2; \quad y(0) = 1 \tag{19}$$

The analytical solution of (19) is obtained as

$$y(x) = \frac{1}{1-x}. \tag{20}$$

This problem was considered by the author in [13]. The logarithm of absolute errors for the solutions obtained are compared with the methods discussed in [13] as given in Figure 4.

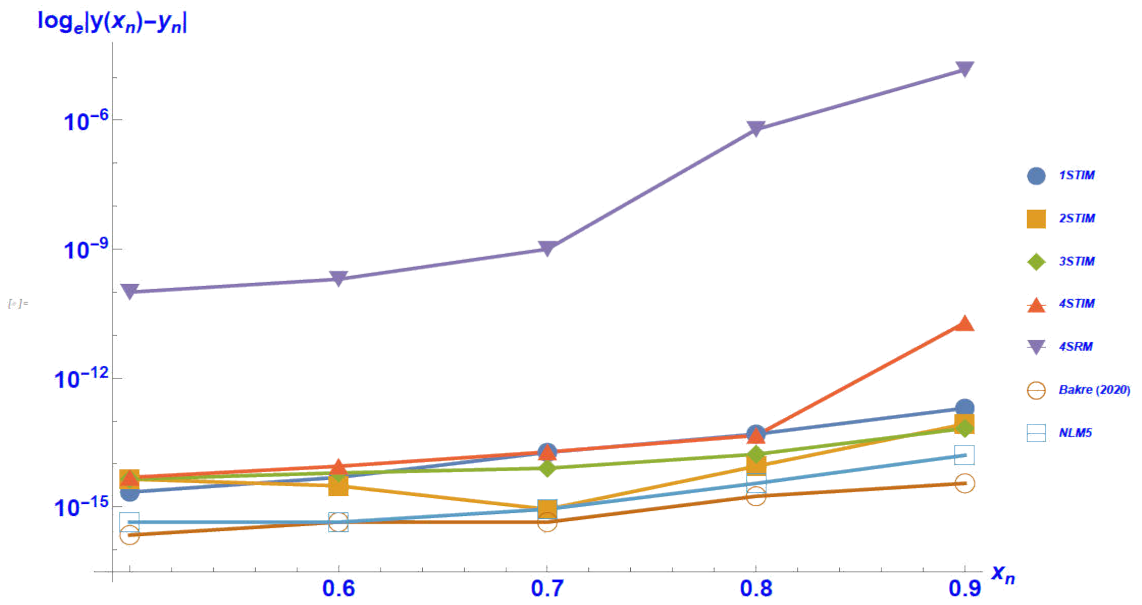

FIGURE 4: Logarithm of absolute errors for the solutions of (19) with step-size h = 0.1.

CONCLUSION

The sixth-order explicit single-step nonlinear method constructed in this work is consistent and absolutely stable. Its region of absolute stability is larger than those of the methods discussed in the literature. The method gave more accurate result on the standard test problems compared with other methods discussed.

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